QUADRATIC POLYNOMIALS AND COMBINATORICS OF THE PRINCIPAL NEST

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Abstract. *The definition of principal nest is supplemented with a system of frames that make possible the classification of combinatorial types for every level of the nest. As a consequence, we give necessary and sufficient conditions for the admissibility of a type and prove that given a sequence of non-renormalizable finite admissible types, there is a quadratic polynomial whose nest realizes the sequence.*

1. Introduction

We will study the combinatorial behavior of the dynamics for quadratic polynomials with (non-periodic) recurrent critical orbit; these are the maps that have a well defined *principal nest*.

In [L2], M. Lyubich developed the principal nest as a tool to provide some examples of infinitely renormalizable parameters at which the Mandelbrot set is locally connected. The nest consists of a subsequence of central puzzle pieces, each determined by the first return of the critical orbit to the preceding nest piece.

As described in Section 2, the principal nest may include non-central pieces at some levels. Each piece V of the nest has a first return map onto the central piece of previous level that contains V.

When the polynomial is real, the lateral pieces of the nest can only be located to the left or right of the central piece. This information, together with the sign of the derivative of the first return maps, is enough to provide a complete classification of real nest types (see [L1]). However, in the complex case, lateral pieces may "hang" from different branches of the Julia set. We exploit this underlying structure to construct a *frame system* that encodes the configuration of the nest. This allows us to describe the possible itineraries of the critical orbit as it visits different levels.

Our main classification result is the following:

Theorem: *Any infinite sequence of finite, weak combinatorial types is realized in the quadratic family, as long as the types satisfy the admissibility condition at every level. The set of parameters that display this sequence of types can be described as the residual intersection in an infinite family of sequences of nested parapieces.*

We illustrate the applicability of frames with a description of maximal hyperbolic components of the Mandelbrot set, and with the construction of complex analogues of the *rotation-like maps* of [BKP]. Further applications, including a classification of complex quadratic Fibonacci maps, are contained in [P].

1.1. **Background and organization.** The concept of a puzzle partition was introduced in [BH1] and [BH2] to study the topology of cubic Julia sets as a function of the critical points. In the late 80's, J.-C. Yoccoz implemented the puzzle in the setting of quadratic polynomials, in order to prove the MLC conjecture for the case of finitely renormalizable parameters (see [H]). The idea of the puzzle construction is to show that the pieces around the critical point become arbitrarily small, thus providing a system of neighborhoods that satisfy the local connectivity condition. For Yoccoz's puzzle, this is done by showing that the moduli of annuli between consecutive pieces generate a divergent series. In the case of the principal nest, the moduli between consecutive nest pieces

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increase in an essentially linear fashion. The principal nest technique underlies Lyubich's proofs of the Feigenbaum-Collet-Tresser conjecture and the theorem on the measure-theoretic attractor.

In order to fix notation, we introduce basic notions of Complex Dynamics in Section 2. In particular, we describe the puzzle construction of Yoccoz and the principal nest following Lyubich.

In Section 3 we define the frame associated to a nest. The construction requires particular care at the initial steps in order to ensure that nest levels and frame levels go hand by hand. Then we specify a labeling of frame cells and produce a language to describe admissible combinatorial types. Our main result (Theorem 3.6 and Corollary 3.7) is stated and proved there.

Section 4 illustrates the use of our construction with two examples; a classification of *maximal* hyperbolic components of the Mandelbrot set according to the combinatorial type of their nests, and an extension of the family of *rotation-like maps* described in [BKP].

A brief summary of holomorphic motions is included in an appendix.

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2. Basics in Complex Dynamics

2.1. **Basic notions.** In order to fix notation, let us start by defining the basic notions of complex dynamics that will be used; we refer the reader to [DH1] and [M1] for details on this introductory material.

We focus attention on the *quadratic family* $\mathcal{Q} := \big\{ f_c : z \mapsto z^2 + c \mid c \in \mathbb{C} \big\}$. For every c, the compact sets $K_c := \big\{ z \mid \text{the sequence } \{f_c^{\circ n}(z)\} \text{ is bounded} \big\}$ and $J_c := \partial K_c$ are called the **filled Julia set** and **Julia set** respectively. Depending on whether the orbit of the critical point 0 is bounded or not, J_c and K_c are connected or totally disconnected. The **Mandelbrot set** is defined as $M := \big\{ c \mid c \in K_c \big\}$; that is, the set of parameters with bounded critical orbit; see Figure 1.

A component of $\operatorname{int} M$ that contains a superattracting parameter will be called a **hyperbolic component**[1]. The boundary of a hyperbolic component can either be real analytic, or fail to be so at one cusp point. The later kind are called **primitive** components. In particular, the hyperbolic component $\heartsuit$ associated to $z \mapsto z^2$ is bounded by a cardioid known as the **main cardioid**.

M contains infinitely many small homeomorphic copies of itself, accumulating densely around ∂M. In fact, every hyperbolic component H other than the main one is the base of one such small copy M'. H is called **prime** if it is not contained in any other small copy. To simplify later statements, prime components are further subdivided in **immediate** (non-primitive components that share a boundary point with $\heartsuit$) and **maximal** (primitive components away from $\partial\heartsuit$).

2.2. **External rays, wakes and limbs.** Since $f_c^{-1}(\infty) = \{\infty\}$, the point ∞ is a fixed critical point and a result of Böttcher yields a change of coordinates that conjugates f_c to $z \mapsto z^2$ in a neighborhood of ∞. With the requirement that the derivative at ∞ is 1, this conjugating map is denoted $\varphi_c : N_c \longrightarrow \overline{\mathbb{C}} \setminus \overline{\mathbb{D}_R}$, where $\mathbb{D}_R$ is the disk of radius $R \geq 1$ and N_c is the maximal domain of unimodality for φ_c. It can be shown that $N_c = \overline{\mathbb{C}} \setminus K_c$ and $R = 1$ whenever $c \in M$. Otherwise, N_c is the exterior of a figure 8 curve that is real analytic and symmetric with respect to 0. In this case, $R > 1$ and K_c is contained in the two bounded regions determined by the 8 curve.

Consider the system of radial lines and concentric circles in $\mathbb{C} \setminus \mathbb{D}_R$ that characterizes polar coordinates. The pull back of these curves by φ_c, creates a collection of **external rays** r_θ $\big(\theta \in [0, 1)\big)$ and **equipotential curves** e_s $\big($here $s \in (R, \infty)$ is called the **radius** of $e_s\big)$ on N_c. These form

[1] Though, of course, it is conjectured that all interior components are hyperbolic.

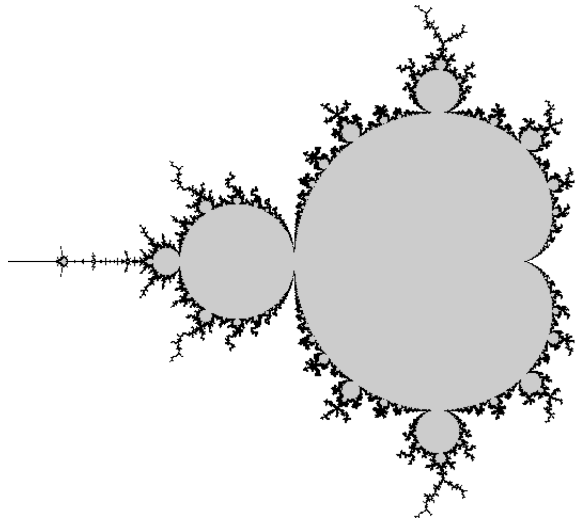

FIGURE 1. *The Mandelbrot set.*

two orthogonal foliations that behave nicely under dynamics: $f_c(r_\theta) = r_{2\theta}$, $f_c(e_s) = e_{(s^2)}$. When $c \in M$, we say that a ray r_θ **lands** at $z \in J_c$ if z is the only point of accumulation of r_θ on J_c.

A similar coordinate system exists around the Mandelbrot set. For $c \notin M$, we define the map

$$\Phi_M(c) := \varphi_c(c). \tag{2.1}$$

In [DH1] it is shown that $\Phi_M : \overline{\mathbb{C}} \setminus M \longrightarrow \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ is a conformal homeomorphism tangent to the identity at ∞. This yields connectivity of M and allows us to define **parametric external rays** and **parametric equipotentials** as in the dynamical case. Since there is little risk of confusion, we will use the same notation (r_θ, e_s) to denote these curves and say that a parametric ray lands at a point $c \in \partial M$ if c is the only point of accumulation of the ray on M.

For the rest of this work, all rays considered, whether in dynamical or parameter plane, will have rational angles. These are enough to work out our combinatorial constructions and satisfy rather neat properties.

Proposition 2.1. *(*[M1]*, ch.18) Both in the parametric and the dynamical situations, if $\theta \in \mathbb{Q}$ the external ray r_θ lands. In the dynamical case, the landing point is (pre-)periodic with the period and preperiod determined by the binary expansion of θ. A point in J_c (respectively ∂M) can be the landing point of at most, a finite number of rays (respectively parametric rays). If this number is larger than 1, each component of the plane split by the landing rays will intersect J_c (respectively ∂M).*

Unless $c = \frac{1}{4}$, f_c has two distinct fixed points. If $c \in M$, these can be distinguished since one of them is always the landing point of the ray r_0. We call this fixed point β. The second fixed point is called α and can be attracting, indifferent or repelling, depending on whether the parameter c belongs to $\heartsuit$, $\partial\heartsuit$, or $\mathbb{C} \setminus \overline{\heartsuit}$. The map $\psi_0 : \heartsuit \longrightarrow \mathbb{D}$ given by $c \mapsto f_c'(\alpha_c)$ is the Riemann map of $\heartsuit$ normalized by $\psi_0(0) = 0$ and $\psi_0'(0) > 0$. Since the cardioid is a real analytic curve except at $\frac{1}{4}$, ψ_0 extends to $\overline{\heartsuit}$.

The fixed point α is parabolic exactly at parameters $c_\eta \in \partial\heartsuit$ of the form $c_\eta = \psi_0^{-1}\left(e^{2\pi i \eta}\right)$ where $\eta \in \mathbb{Q} \cap [0, 1)$. If $\eta \neq 0$, c_η is the landing point of two parametric rays $r_{t^-(\eta)}$ and $r_{t^+(\eta)}$.

Definition: The closure of the component of $\mathbb{C} \setminus \left(r_{t^-(\eta)} \cup c_\eta \cup r_{t^+(\eta)}\right)$ that does not contain $\heartsuit$ is called the **η-wake** of M and is denoted W_η. The **η-limb** is defined as $L_\eta = M \cap W_\eta$.

Definition: Say that $\eta = \frac{p}{q}$, written in lowest terms. Then $\mathcal{P}\left(\frac{p}{q}\right)$ will denote the unique set of angles whose behavior under doubling is a cyclic permutation with combinatorial rotation number $\frac{p}{q}$.

If $\mathcal{P}\left(\frac{p}{q}\right) = \{t_1, \ldots, t_q\}$, then for any parameter $c \in L_{p/q}$ the corresponding point α splits K_c in q parts, separated by the q rays $\{r_{t_1}, \ldots, r_{t_q}\}$ landing at α. The two rays whose angles span the shortest arc separate the critical point 0 from the critical value c; these two angles turn out to be $t^-(\frac{p}{q})$ and $t^+(\frac{p}{q})$.

2.3. **Yoccoz puzzles.** The Yoccoz **puzzle** is well defined for parameters $c \in L_{p/q}$ for any any $\frac{p}{q} \in \mathbb{Q} \cap [0,1)$ with $(p,q) = 1$. If 0 is not a preimage of α, the puzzle is defined at infinitely many depths and we will restrict attention to these parameters. Since we describe properties of a general parameter, we will omit the subscript and write f instead of f_c, K instead of K_c and so on.

Let us fix the neighborhood U of K bounded by the equipotential of radius 2. The rays that land at α determine a partition of $U \setminus \{r_{t_1}, \ldots, r_{t_q}\}$ in q connected components. We will call the closures $Y_0^{(0)}, Y_1^{(0)}, \ldots, Y_{q-1}^{(0)}$ of these components, **puzzle pieces** of depth 0. At this stage the labeling is chosen so that $0 \in Y_0^{(0)}$ and $f\left(K \cap Y_j^{(0)}\right) = K \cap Y_{j+1}^{(0)}$; where the subindices are understood as residues modulo q. In particular, $Y_1^{(0)}$ contains the critical value c and the angles of its bounding rays are $t^-(\frac{p}{q}), t^+(\frac{p}{q})$.

The puzzle pieces $Y_i^{(n)}$ of higher depths are recursively defined as the closures of every connected component in $f^{\circ(-n)}\left(\bigcup \operatorname{int} Y_j^{(0)}\right)$; see Figure 2. At each depth n, there is a unique piece which contains the critical point and we will always choose the indices so that $0 \in Y_0^{(n)}$.

We will denote by P_n the collection of pieces of level n. The resulting family $\mathcal{Y}_c := \{P_0, P_1, \ldots\}$ of puzzle pieces of all depths, has the following two properties:

- **P1** Any two puzzle pieces either are nested (with the piece of higher depth contained in the piece of lower depth), or have disjoint interiors.
- **P2** The image of any piece $Y_j^{(n)}$ $(n \geq 1)$ is a piece $Y_i^{(n-1)}$ of the previous depth $n-1$. The restricted map $f : \operatorname{int} Y_j^{(n)} \longrightarrow \operatorname{int} Y_i^{(n-1)}$ is a 2 to 1 branched covering or a conformal homeomorphism, depending on whether $j = 0$ or not.

These properties characterize $\mathcal{Y}_c$ as a Markov family, endowing the puzzle partition with dynamical meaning.

Note that the collection of ray angles at depth n consists of all n-preimages of $\{r_{t_1}, \ldots, r_{t_q}\}$ under angle doubling. The union of all pieces of depth n is the region enclosed by the equipotential $e_{\left(2^{2^{-n}}\right)}$. Note also that every piece Y of depth n is the n^{th} preimage of some piece of level 0. By further iteration, Y will map onto a region determined by the same rays as $Y_0^{(0)}$ and a possibly larger equipotential. This provides a 1 to 1 correspondence between puzzle pieces and preimages of 0. The distinguished point inside each piece is called the **center** of the piece.

2.4. **Adjacency Graphs.** Given a set of puzzle pieces $P \subset P_n$, we define the **dual graph** $\Gamma(P)$ as a formal graph whose set of vertices is P and whose edges join pairs of pieces that share an arc of external ray. It is always possible to produce an isomorphic model of $\Gamma(P)$ sitting in the plane, without intersecting edges and such that it respects the natural immersion of $\Gamma(P)$ in the plane.

Definition: When $P = P_n$, we call $\Gamma_n := \Gamma(P_n)$ the **puzzle graph** of depth n. In this context, the vertices corresponding to the central piece $Y_0^{(n)}$ and the piece around the critical value $f_c(0)$ are denoted ξ_n and η_n respectively.

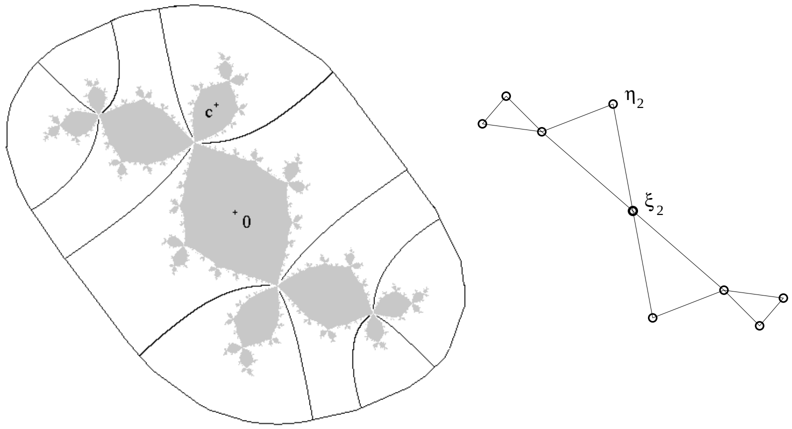

FIGURE 2. *Puzzle of depth 2 and its corresponding graph. Splitting the graph at ξ_2 we obtain the graphs $Puzz_2^-$ and $Puzz_2^+$; both shaped like a bow tie and isomorphic to Γ_1.*

Definition: The vertices ξ_n and η_n determine two partial orders on the vertex set of Γ_n as follows: If $a, b \in V(\Gamma_n)$, we write $a \succ_{\eta_n} b$ when every path from a to η_n passes through b. We write $a \succ_{\xi_n} b$ when every path from a to ξ_n passes through b or through its symmetric image with respect to the origin.

The following are natural consequences of the definitions; see Figure 2 for reference.

Proposition 2.2. *The puzzle graphs of f satisfy:*

- **G1** *Γ_n has 2-fold central symmetry around ξ_n.*
- **G2** *Γ_0 is a q-gon whenever $c \in L_{p/q}$. For $n \geq 1$, Γ_n consists of 2^n q-gons linked at their vertices in a tree-like structure; i.e. the only cycles on this graph are the q-gons themselves.*
- **G3** *For $n \geq 1$, removing ξ_n and its edges splits Γ_n into 2 disjoint (possibly disconnected) isomorphic graphs. Reattaching ξ_n to each, and adding the corresponding edges defines the connected graphs $Puzz_n^-$ and $Puzz_n^+$ (here, $\eta_n \in Puzz_n^-$). Then $\Gamma_n = Puzz_n^- \cup Puzz_n^+$ and $Puzz_n^-$, $Puzz_n^+$ are isomorphic to Γ_{n-1} with $\mp\eta_n$ playing the role of ξ_{n-1} in $Puzz_n^\pm$.*
- **G4** *For $n \geq 1$ there are two natural maps: $f^* : \Gamma_n \longrightarrow \Gamma_{n-1}$ induced by f, and $\iota^* : \Gamma_n \longrightarrow \Gamma_{n-1}$ induced by the inclusion among pieces of consecutive depths. f^* is 2 to 1 except at ξ_n and sends $Puzz_n^\pm$ onto Γ_{n-1}. In turn, ι^* collapses the outermost q-gons into vertices.*
- **G5** *The map $f^* : (\Gamma_n, \succ_{\xi_n}) \longrightarrow (\Gamma_{n-1}, \succ_{\eta_{n-1}})$ respects order. That is, if $a \succ_{\xi_n} b$ then $f^*(a) \succ_{\eta_{n-1}} f^*(b)$.*

Definition: Let Γ be a graph isomorphic to a subgraph of Γ_n and Γ' a graph isomorphic to a subgraph of Γ_{n-1}. A map $E : \Gamma \longrightarrow \Gamma'$ that satisfies **G1** and **G2** will be called **admissible** if it also respects order in the sense of **G5**.

Proof of Proposition 2.2: Property **G1** and the existence of f^* and ι^* are immediate consequences of the structure of quadratic Julia sets. The configuration of Γ_0 is given by the rotation number around α and then the tree-like structure of Γ_n ($n \geq 1$) follows from **G3**.

Consider a centrally symmetric simple curve $\gamma \subset Y_0^{(n)}$ connecting two opposite points of the equipotential curve $e_{(2^{2-n})}$ that bounds $Y_0^{(n)}$. Then γ splits the simply connected region $\bigcup_{Y \in P_n} Y$ in 2 identical parts. Therefore, $\Gamma \setminus \xi_n$ is formed by 2 disjoint graphs justifying the existence of $\mathrm{Puzz}_n^\pm$. However, $\partial Y_0^{(n)}$ may contain several segments of $e_{(2^{2-n})}$; so γ, and consequently $\mathrm{Puzz}_n^\pm$,

are not uniquely determined. This ambiguity is not consequential; Lemmas 3.4 and 3.5 describe the proper method of handling it.

The fact that f maps the central piece to a non-central one containing the critical value legitimizes the selection of Puzz_n^- as the unique graph containing η_n. By symmetry, every piece of P_n except the central one has a symmetric partner and they both map in a 1 to 1 fashion to the same piece of P_{n-1}. The isomorphisms in **G3** follow.

If two pieces A, B of depth n share a boundary ray, their images will too. Moreover, letting A', B' be the pieces of depth $n-1$ containing A and B, it is clear that $\partial A'$ and $\partial B'$ must share the *same* ray as ∂A and ∂B. This shows that f^* and ι^* effectively preserve edges and are well defined graph maps. Clearly f^* is 2 to 1, so to complete the proof of **G4** we only need to justify the collapsing property of ι^*, and by Property **G3**, it is sufficient to consider the case $\iota^* : \Gamma_1 \longrightarrow \Gamma_0$. Now, the non-critical piece $Y_j^{(0)}$ contains a unique piece Y_j of P_1. However, the critical piece $Y_0^{(0)}$ contains a total of q different pieces of depth 1: a smaller central piece $Y_0^{(1)}$ and $q-1$ lateral pieces $-Y_j$. The resulting graph, Γ_1, consists then of two q-gons joined at the vertex ξ_1. Under ι^*, one of these q-gons collapses on the critical vertex ξ_0.

To prove **G5**, let us construct the tree Γ'_n with 2 to 1 central symmetry by collapsing every q-gon into a single vertex. The orders $\succ_{\xi'_n}$, $\succ_{\eta'_n}$ in Γ'_n are induced by the orders in Γ_n. Then the corresponding map $f^{*\prime} : \left(\Gamma'_n, \succ_{\xi_n}\right) \longrightarrow \left(\Gamma'_{n-1}, \succ_{\eta_n}\right)$ is a 2 to 1 map on trees that takes each half of Γ'_n injectively into a sub-tree of Γ'_{n-1} and respects order. Since vertices in a cycle are not ordered, f^* respects order as well. $\square$

2.5. **Parapuzzle.** While the puzzle encodes the combinatorial behavior of the critical orbit for a specific map f_c, the *parapuzzle* dissects the parameter plane into regions of parameters that share similar behaviors: In every wake of M we define a partition in pieces of increasing depths, with the property that all parameters inside a given *parapiece* share the same critical orbit pattern up to a specific depth.

Definition: Consider a wake $W_{p/q}$ and let $n \geq 0$ be given. Call W^n the wake $W_{p/q}$ truncated by the equipotential $e_{\left(2^{2-n}\right)}$ and consider the set of angles $\mathcal{P}_n(\frac{p}{q}) = \left\{t \mid 2^n t \in \mathcal{P}\left(\frac{p}{q}\right)\right\}$ (compare Subsection 2.2). The **parapieces** of $W_{p/q}$ at depth n are the closures of the components of $W^n \setminus \left\{r_t \mid t \in \mathcal{P}_n(\frac{p}{q})\right\}$.

Note: Even though the critical value $f_c(0)$ is simply c, it will be convenient to write $c \in \Delta$ when Δ is a parapiece and $f_c(0) \in V$ when V is a piece in the dynamical plane of f_c. In general, we will use the notation $\mathrm{OBJ}[c]$ to refer to dynamically defined objects OBJ associated to a specific parameter c.

Definition: When the boundary of a dynamical piece A is described by the same equipotential and ray angles as those of a parapiece B, we denote this relation by $\partial A \mathrel{\hat{=}} \partial B$.

Definition: Let $c \in M$ be a parameter whose puzzle is defined up to depth n. We denote by $\mathrm{CV}_n[c] \in P_n[c]$ the piece of depth n that contains the critical value: $f_c(0) \in \mathrm{CV}_n[c]$.

A consequence of Formula 2.1 is the well known fact that follows. For a proof of the main statement, refer to [DH2] or [R]. For a proof of the winding number property, refer to [D2] and Proposition 3.3 of [L3]; also, see the Appendix for the definition of holomorphic motions.

Proposition 2.3. *Let Δ be a parapiece of depth n in some wake W. Then $CV_n[c] \mathrel{\hat{=}} \Delta$ for every $c \in \Delta$ so the family $\left\{c \mapsto CV_n[c] \mid c \in \Delta\right\}$ is well defined; it determines a holomorphic motion of the critical value pieces. The holomorphic motion has $\left\{c \mapsto f_c(0)\right\}$ as a section with winding number 1.*

We can interpret the result on winding number as loosely saying that, as c goes once around $\partial\Delta$, the critical value $f_c(0)$ goes once around $\partial\mathrm{CV}_n$. However, this description is not entirely accurate since $\partial\mathrm{CV}_n[c]$ changes with c.

Let us mention the following examples of combinatorial properties that depend on the behavior of the first n iterates of 0. The fact that these entities remain unchanged for $c \in \Delta$ follows from Proposition 2.3 and will be useful in the next sections.

- The isomorphism type of $\Gamma_n[c]$.
- The combinatorial boundary of every piece of depth $\leq n$.
- The location within $P_n[c]$ of the first n iterates of the critical orbit.

From the general results of [L3], we can say more about the geometric objects associated to the above examples.

Proposition 2.4. *Each of the sets listed below moves holomorphically as c varies in Δ:*

- *The boundary of every piece of depth $\leq n$.*
- *The first n iterates of the critical orbit.*
- *The collection of j-fold preimages of α and β ($j \leq n$).*

2.6. **Principal nest.** The principal nest is well defined for parameters c that belong neither to $\overline{\heartsuit}$ nor to an immediate component. The first condition means that both fixed points are repelling (so the puzzle is defined), while the second condition characterizes those polynomials that do not admit an *immediate renormalization* as described below. We restrict further to parameters c such that the orbit of 0 is recurrent to ensure that the nest is infinite. These necessary conditions will justify themselves as we describe the nest.

In order to explain the construction of the principal nest, we need a more detailed description of the puzzle partition at depth 1 (use Figure 3 for reference). As a note of warning, the pieces of depth 1 will be renamed to reflect certain properties of P_1. That is, we will override the use of the symbols $Y_j^{(1)}$.

The puzzle depth P_1 consists of $2q-1$ pieces of which $q-1$ are the restriction to lower equipotential of the pieces $Y_1^{(0)}, Y_2^{(0)}, \ldots, Y_{q-1}^{(0)}$. Such pieces cluster around α and will be denoted $Y_1, Y_2, \ldots, Y_{q-1}$. The restriction of $Y_0^{(0)}$ however, is further divided into the union of the critical piece $Y_0^{(1)}$ and $q-1$ pieces $Z_1, Z_2, \ldots, Z_{q-1}$ which are symmetric to the corresponding Y_j and cluster around $-\alpha$. The indices are again determined by the rotation number of α so that $f(Z_j)$ is opposite to Y_j and consequently $f(Z_j) = Y_{j+1}^{(0)}$.

Note that $f^{\circ q}(0) \in Y_0^{(0)}$, so we face two possibilities. It may happen that $f^{\circ jq}(0) \in Y_0^{(1)}$ for all j, in which case we can find *thickenings* of $Y_0^{(1)}$ and $Y_0^{(0)}$, that yield the **immediate renormalization** $f^{\circ q} : Y_0^{(1)} \longrightarrow Y_0^{(0)}$ described by Douady and Hubbard; or else, we can find the least k for which the orbit of 0 under $f^{\circ q}$ escapes from $Y_0^{(1)}$. We will assume that this is the case, so $f^{\circ kq}(0) \in Z_\nu$ for some ν and we call kq the **first escape time.**

The initial nest piece V_0^0 is defined as the (kq)-fold pull back of Z_ν along the critical orbit; that is, the unique piece that satisfies $0 \in V_0^0$ and $f^{\circ kq}(V_0^0) = Z_\nu$. In fact, V_0^0 can also be defined as **the largest central piece that is compactly contained in** $Y_0^{(1)}$: Notice that $Z_\nu \Subset Y_0^{(0)}$ so $V_0^0 \Subset Y_0^{(1)}$; that is, $\left(\operatorname{int} Y_0^{(1)}\right) \setminus V_0^0$ is a non-degenerate annulus.

The higher levels of the principal nest are defined inductively. Suppose that the pieces $V_0^0, V_0^1, \ldots, V_0^n$ have been already constructed. If the critical orbit never returns to V_0^n then the nest is finite. Otherwise, there is a first return time ℓ_n such that $f^{\circ \ell_n}(0) \in V_0^n$; then we define V_0^{n+1} as the *critical* piece that maps to V_0^n under $f^{\circ \ell_n}$.

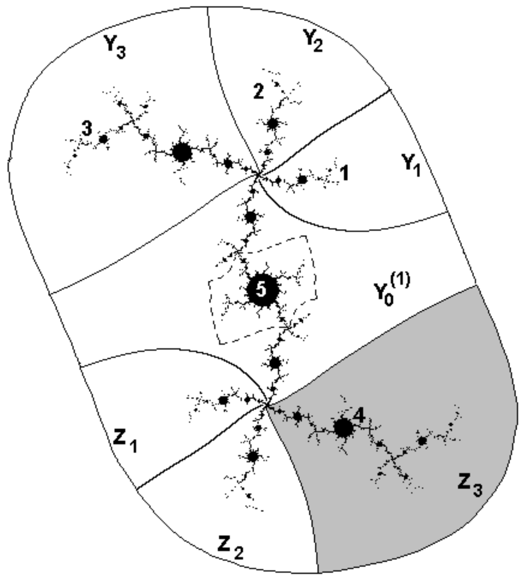

FIGURE 3. *Puzzle $P_1(f_c)$ of depth 1, where $c = (0.35926...) + i(0.64251...)$ is the center of the component of period 5 in $L_{1/4}$. The first escape is $f_c^{\circ 4}(0) \in Z_3$ and the pull back V_0^0 is shown in dotted lines. Note that $f^{\circ 5}(0) \in V_0^0$. This creates at once the piece $V_0^1 \Subset V_0^0$ around the central component of $\mathbb{C} \setminus J_c$ (V_0^1 is not shown).*

Proposition 2.5. *The principal nest $V_0^0 \Supset V_0^1 \Supset \ldots$ is a family of strictly nested pieces centered around 0.*

Proof: V_0^0 is a piece of depth kq (the first escape time). Since V_0^1 is a $f^{\circ \ell_1}$-pull back of V_0^0, it is a piece of depth $kq + \ell_1$ and, in general, V_0^n will be a piece of depth $kq + \ell_1 + \cdots + \ell_n$. Since all pieces contain 0, Property **P1** implies that $V_0^j \supset V_0^{j+1}$.

Recall that $V_0^0 \Subset Y_0^{(1)}$; thus, the $f^{\circ \ell_1}$-pull backs of these 2 pieces satisfy $V_0^1 \Subset X$ with X a central piece of depth $1 + \ell_1$. Now, $0 \notin Z_\nu$, so $f^{\circ kq}(0)$ requires further iteration to reach a central piece; i.e., $\ell_1 > kq$. By construction, V_0^0 is a central piece of depth $1 + kq$, so Property **P1** implies $V_0^1 \Subset X \subset V_0^0$. An analogous argument yields the strict nesting property for the nest pieces of higher depth. $\square$

Definition: The **principal annuli** $V_0^{n-1} \setminus V_0^n$ will be denoted A_n.

It may happen that $\ell_{n+1} = \ell_n$; this means that not only does 0 return to V_0^n under $f^{\circ \ell_n}$, but even deeper to V_0^{n+1} without further iteration. In this case we say that the return is **central** and we call a chain of consecutive central returns $\ell_n = \ell_{n+1} = \ldots = \ell_{n+s}$ a **cascade of central returns**. An infinite cascade means that the sequence $\{\ell_n\}$ is eventually constant, so $f^{\circ \ell_n}(0) \in \bigcap_{j=n}^{\infty} V_0^j$. By definition, $f^{\circ \ell_n} : V_0^{n+1} \longrightarrow V_0^n$ is a **renormalization** of f; that is, a 2 to 1 branched cover of V_0^n such that the orbit of the critical point is defined for all iterates.

The return to V_0^n, however, can be non-central. In fact, it is possible to have several returns to V_0^n before the critical orbit hits V_0^{n+1} for the first time. When a return is non-central, the description of the nest at that level is completed by the introduction of the **lateral** pieces $V_k^n \in V_0^{n-1} \setminus V_0^n$. Let $\mathcal{O} \subset K$ denote the critical orbit $\mathcal{O} = \left\{ f^{\circ j}(0) | j \geq 0 \right\}$ and take a point $z \in \overline{\mathcal{O}} \cap V_0^{n-1}$ whose forward orbit returns to V_0^{n-1}. If we call $r_{n-1}(z)$ the first return time of z back to V_0^{n-1}, we can define $V^n(z)$ as the unique puzzle piece that satisfies $z \in V^n(z)$ and $f^{\circ r_{n-1}(z)}\big(V^n(z)\big) = V_0^{n-1}$.

In particular, it is clear that $V^n(0)$ is just the same as V_0^n and that any 2 pieces created by this process are disjoint or equal.

Definition: The collection of all pieces $V^n(z)$ for $z \in \overline{\mathcal{O}} \cap V_0^{n-1}$ that actually contain a point of $\mathcal{O}$ is denoted $\mathcal{V}^n$ and referred to as the **level** n of the nest.

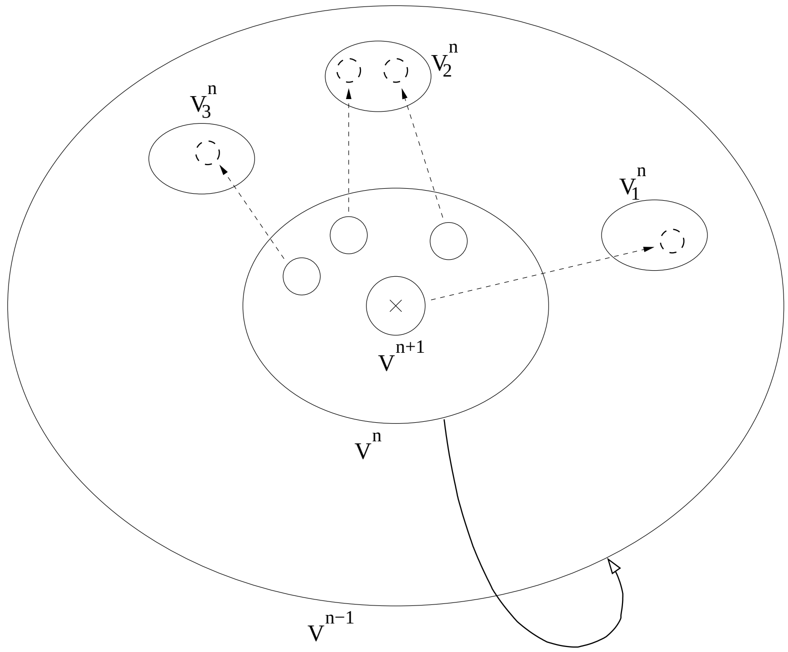

FIGURE 4. *Relation between consecutive nest levels. The curved arrow represents the first return map $f^{\circ \ell_n} : V_0^n \longrightarrow V_0^{n-1}$ which is 2 to 1. The dotted arrows show a possible effect of this map on each nest piece of level $n+1$. Each V_j^{n+1} may require a different number of additional iterates to return to this level and map onto V_0^n.*

Under the assumption that c is recurrent, the principal nest will have infinitely many levels. Let us assume the parameter c is not periodic. Then it is called **reluctantly recurrent** if for some central piece V_0^n there are arbitrarily long sequences of univalent f_c-pull backs of V_0^n along backward orbits in the postcritical set $\overline{\mathcal{O}}$. Otherwise, c is called **persistently recurrent**.

Lemma 2.6. (see [L1],[Ma]) *If f_c is persistently recurrent, $\overline{\mathcal{O}}$ is a Cantor set and the action of $f_{c|_{\overline{\mathcal{O}}}}$ is minimal. When f_c is not renormalizable, c is reluctantly recurrent if and only if some central piece V_0^n has infinitely many 1 to 2 pull backs along backward orbits of $\mathcal{O}$.*

Observation: In particular, if c is non-renormalizable but every level of the principal nest has a finite number of pieces, then f_c acts minimally on the postcritical set. In this situation, we can name the pieces $\mathcal{V}^n = \{V_0^n, V_1^n, \ldots, V_{m_n}^n\}$ in such a way that the first visit of the critical orbit to V_i^n occurs before the first visit to V_j^n whenever $i < j$. Obviously, the value of $r_{n-1}(z)$ is independent of $z \in V_k^n$; thus we will denote it $r_{n,k}$.

Definition: For finite $\mathcal{V}^n$ we define the map:

$$g_n : \bigcup_{\mathcal{V}^n} V_k^n \longrightarrow V_0^{n-1},$$

given on each V_k^n by $g_{n|_{V_k^n}} \equiv f^{\circ r_{n,k}}$.

The map g_n satisfies the properties of a *generalized quadratic-like* (**gql**) *map,* i.e.:

- $|\mathcal{V}^n| < \infty$.
- $\bigcup_{\mathcal{V}^n} V_k^n \Subset V_0^{n-1}$ and all the pieces of $\mathcal{V}^n$ are pairwise disjoint.
- $g_{n|_{V_k^n}} : V_k^n \longrightarrow V_0^{n-1}$ is a 2 to 1 branched cover or a conformal homeomorphism depending on whether $k = 0$ or not.

Note that g_n usually is the result of a different number of iterates of f when restricted to different V_k^n. However, since we often refer to the map g_n as acting on individual pieces, it is typographically convenient to introduce the notation

Definition: The map $g_{n|_{V_k^n}} = f^{\circ r_{n,k}}$ will be denoted $g_{n,k}$.

Thus, $g_{n,k}(V_k^n) = V_0^{n-1}$ is a 2 to 1 branched cover or a homeomorphism depending on whether $k = 0$ or not.

From this moment on, we will assume that the principal nest is infinite, and that f is non-renormalizable, thus excluding the possibility of an infinite cascade of central returns. In this situation we say that f is **combinatorially recurrent.**

2.7. **Paranest.** The *paranest* is well defined around parameters c outside the main cardioid that are neither immediately renormalizable nor postcritically finite.

Definition: If c is a parameter such that f_c has a well defined nest up to level n (for $n \geq 0$), the **paranest** piece $\Delta^n[c]$ is defined by the condition $\partial\Delta^n[c] \stackrel{\circ}{=} \partial f_c(V_0^n)$; where V_0^n is the central piece of level n in the principal nest of f_c. By the Douady-Hubbard theory, $\Delta^n[c]$ is a well defined region.

The definition of principal nest, together with Proposition 2.3 imply that when $c' \in \Delta^n[c]$, the principal nests of f_c and $f_{c'}$ are identical until the first return $g_n(0)$ to V_0^{n-1} (which creates V_0^n). In fact, the relevant pieces move holomorphically as c' varies and $\Delta^n[c]$ is the largest parameter region over which the initial set of ℓ_n iterates of 0 (recall that $g_n \equiv f^{\circ \ell_n}$) moves holomorphically without crossing piece boundaries.

Following the presentation of [L3], the family $\{g_n[c'] : V_0^n[c'] \longrightarrow V_0^{n-1}[c'] \mid c' \in \Delta^n[c]\}$ is a proper DH quadratic-like family with winding number 1. The last property follows from Proposition 2.3 since g_n is the first return to a critical piece at this level.

Since the central nest pieces are strictly nested, the above definition implies that the pieces of the paranest are strictly nested as well. It follows that $(\operatorname{int} \Delta^n) \setminus \Delta^{n-1}$ is a non-degenerate annulus. One of the main concerns is to estimate its modulus or, as it is sometimes called, the **paramodulus.**

3. Frame system

Let f_c have an infinite principal nest. For real parameters, Lyubich provides in [L1] a complete criterion for compatibility between consecutive nest levels. Since the Julia set is an interval when $c \in \mathbb{R}$, the compatibility conditions are given in terms of the left/right location of lateral pieces (relative to 0) and the orientation of each $g_{n,k}$ (as an interval map).

In the case of a complex parameter, the nest falls short of being a complete invariant for the dynamics of the critical orbit. The reason is that the nest description does not account for the relative positions between lateral pieces. In contrast to the real case, the Julia set of a complex polynomial displays a complicated structure that varies with the parameter. Lateral pieces may be attached to different branches of the Julia set. For this reason, a record of the relative positions of nest pieces must be preceded by a description of the combinatorial structure around them.

In this Section we enhance the principal nest with the addition of a *frame system.* This provides the necessary language to locate the lateral nest pieces and describe as a consequence, the behavior

of the critical orbit. *The idea is to split the central nest pieces in smaller regions by a procedure that resembles the construction of the puzzle.*

For convenience, let us summarize certain aspects of the construction before giving it in detail. Recall that the definition of V_0^0 guarantees that $\big(\operatorname{int} Y_0^{(1)}\big) \setminus V_0^0$ is a non-degenerate annulus. Because of this initial step, and since our purpose is that frame levels correspond to nest levels, we need to pay individual attention to the construction of the first three levels of the frame. Figure 5 illustrates these initial steps. We will keep in mind our convention of distinguishing between puzzle depths and nest levels. Accordingly, frames will be also stratified in levels since their definition depends on the same pull backs as those used for the nest. To distinguish between nest pieces and frame pieces, the latter will be referred to as *cells*. As a final note of warning, we will abuse our notation and use F_n to refer to the frame as well as to the system of curves that bound its cells. In particular, we will use ∂F_n to describe the union of curves that form the boundary of the union of all cells in F_n. The context will always make clear which meaning is intended.

3.1. **Frames.** As mentioned above, some attention must be given to the construction of the frames F_0, F_1 and F_2 so that the properties in Proposition 3.3 hold. Figure 3 provides a useful reference. After this, the frames of higher levels are defined inductively.

Consider the puzzle partition at depth 1 and recall that kq denotes the first escape of the critical orbit to Z_ν. The **initial frame** F_0 is the collection of nest pieces $F_0 = \big\{Y_0^{(1)}\big\} \cup \big\{\bigcup_{j=1}^{q}\{Z_j\}\big\}$, each of which is called a **frame cell**. In particular, $\Gamma(F_0)$ is a q-gon. The frame F_1 is the collection of $f^{\circ kq}$-pull backs of cells in F_0 along the orbit of 0.

From the definition, one of the cells of F_1 is the central piece V_0^0 that maps 2 to 1 onto $Z_\nu \in F_0$. The pull back of any other cell $A \in F_0$ consists of two symmetrically opposite cells, each mapping univalently onto A. We say that F_1 is a *well defined unimodal* pull back of F_0.

Lemma 3.1. *All the cells of F_1 are contained in $Y_0^{(1)}$.*

Proof: Since $kq > 1$, $f^{\circ kq}\big(Y_0^{(1)}\big)$ is an extension of $Y_0^{(0)}$ to a larger equipotential. Thus, $f^{\circ kq}\big(Y_0^{(1)}\big)$ contains all cells of F_0. □

Let λ be the first return time of 0 to a cell of F_1. By Lemma 3.1, the collection F_2' of pull backs of cells in F_1 along the $f^{\circ\lambda}$-orbit of 0 is well defined and 2 to 1. Unfortunately, it does not cover every point of J_f inside V_0^0. We will give first some results about F_2' and define afterward a complete frame of level 2.

Lemma 3.2. *The temporary frame F_2' satisfies:*

(1) *All cells of F_2' are contained in V_0^0.*
(2) *V_0^1 is contained in the central cell of F_2'.*

Proof: First note that $\lambda = kq + (q - \nu)$ is the first return of 0 to $Y_0^{(1)}$ after the first escape to Z_ν. We have $kq < \lambda \le \ell_0$, where the second inequality is true since $V_0^0 \in F_1$. Then the first return to F_1 occurs no later than the first return to V_0^0. By definition, $f^{\circ\lambda}(V_0^0)$ is just $Y_0^{(0)}$ extended to a larger equipotential. Since all cells of F_1 are inside $Y_0^{(1)} \subset f^{\circ\lambda}(V_0^0)$, the first assertion follows.

Now, V_0^1 is central. By the Markov properties of $\mathcal{Y}_c$, either V_0^1 is contained in the central cell C of F_2' or vice versa. However, both $f^{\circ\ell_0}(V_0^1)$ and $f^{\circ\lambda}(C)$ belong to F_1. Since $\ell_0 \ge \lambda$, the first possibility is the one that holds. This proves property (2). □

Our intention is to extend F_2' to a frame that covers the intersection $J_f \cap V_0^0$. To do this, we just need to add the $f^{\circ\lambda}$-pull backs of the pieces Z_ν. The union of those pull backs with the cells of F_2' is the frame F_2.

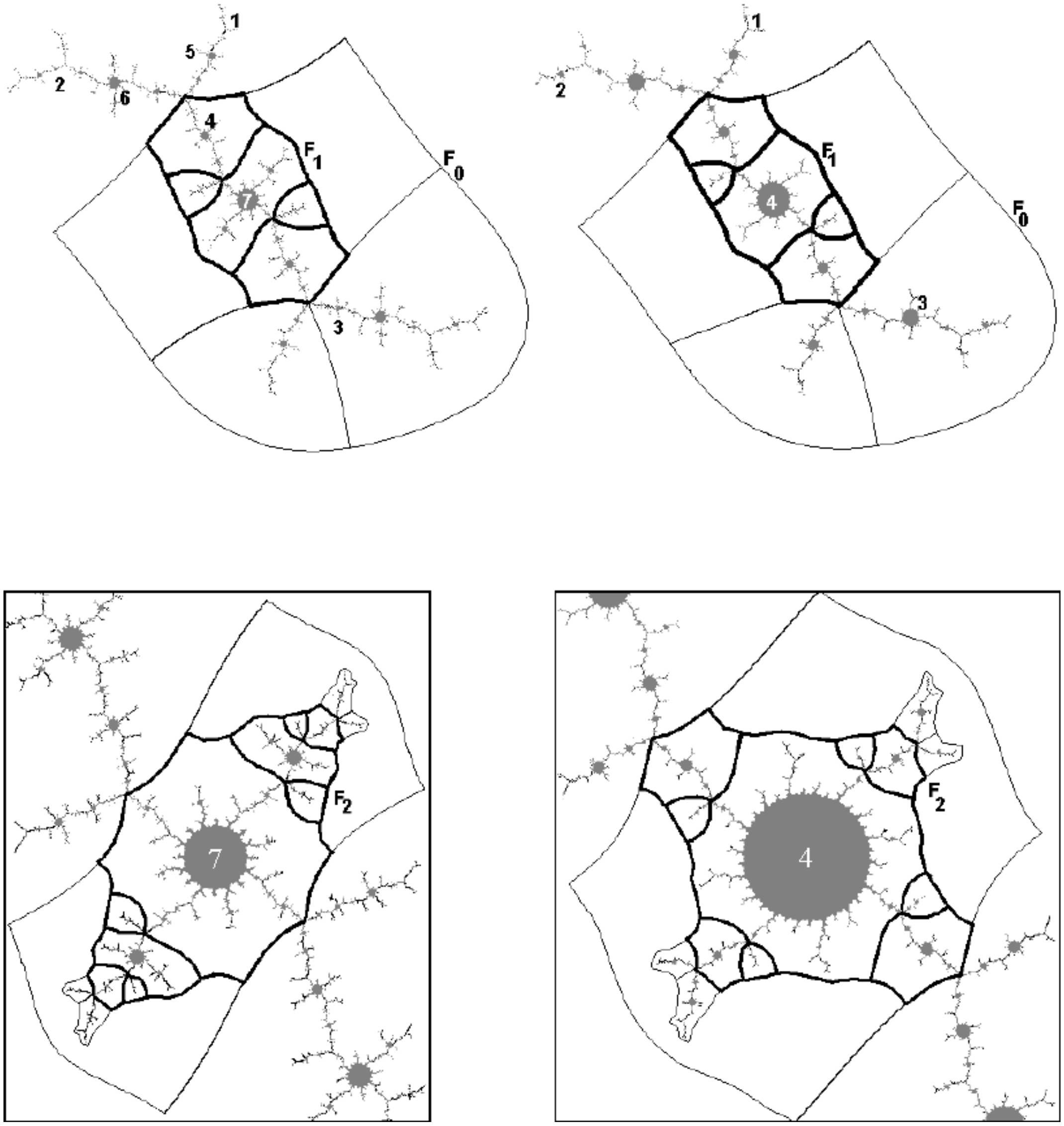

FIGURE 5. *Both of these parameters belong to the left antenna of $L_{1/3}$; they are centers of components of periods 7 and 4. Above we can see that the structures of the frames of levels 0 and 1 coincide between the two examples. Still, the first return to F_1 falls in each case on a different cell, producing dissimilar frames of level 2. The pull back of cells in F_1 produces a preliminary frame F_2', shown in heavy line on the second row. The complete frame F_2, inside V_0^0 has $2(q-1)$ additional cells (here $q = 3$) in order to cover all of $J_f \cap V_0^0$.*

After introducing the first frames and relating them to the initial levels of the nest, we can give the complete definition of the *frame system.* The driving idea of this discussion is that the internal structure of a frame F_{n+2}, represented by the graph $\Gamma(F_{n+2})$, provides a decomposition of $J_f \cap V_0^n$ that helps to describe the combinatorial type of the nest at level $n+1$.

Definition: For $n \geq 0$ consider the first return $g_n(0) \in V_0^n$ and define F_{n+3} as the collection of g_n-pull backs of cells in F_{n+2} along the critical orbit. The family $\mathcal{F}_c = \{F_0, F_1, \ldots\}$ is called a **frame system** for the principal nest of f_c and each piece of a frame is called a **cell**.

The dual graph $\Gamma(F_n)$ (see Subsection 2.4) is called the **frame graph.** As in the case of the puzzle graph, we consider $\Gamma(F_n)$ with its natural embedding in the plane.

Let us mention now some properties of frame systems.

Proposition 3.3. *The frame system satisfies:*

1. *Frames exist at all levels.*
2. *The union of cells $\bigcup_{C_i \in \mathcal{F}_n} C_i$ forms a cover of $K_{f_c} \cap V_0^{n-2}$.*
3. *The central cell of F_n contains the nest piece V_0^{n-1}.*
4. *Each F_n has 2-fold central symmetry around 0.*
5. *Suppose there is a non-central return; then, eventually all nest pieces are compactly contained in cells of the corresponding frame.*

The following observation will help clarify the definition of frames (also, refer to Figure 5). As follows from the comment after Lemma 3.1, the union of cells in $\mathcal{F}_2$ covers exactly the intersection of K_f with the nest piece V_0^0. This is because V_0^0 can be described as the pull back of $Y_0^{(0)}$ under the first return map to F_1. Then, we can think of this union of cells as a single piece, determined by the same rays as V_0^0, but cut off by a lower equipotential.

Proof of Proposition 3.3: F_0 and F_1 are easily seen to exist from their construction. Since F_1 covers the central part of K_f between α and $-\alpha$, there will definitely be a return to it, creating F_2'. As we saw already, this frame is contained inside V_0^0, so its pull backs are well defined as long as there are new levels of the nest. In particular, this already proves claim 2. Since the principal nest is infinite, the critical point is recurrent or the map is renormalizable. Either case creates critical returns to central nest pieces of arbitrarily high level, so F_{n+1} is defined.

The piece V_0^0 is actually the central cell of F_1. Now, the first return to F_1 cannot occur later than the first return to V_0^0, so the central cell C of F_2 is of lower depth than V_0^1; thus, $V_0^1 \subset C$. Afterwards, the depth from V_0^{n-1} to V_0^n increases by ℓ_{n-1}, while the depth from F_n to F_{n+1} increases ℓ_{n-2}. Inductively, since $V_0^{n-1} \subset F_n$ and $\ell_{n-2} \leq \ell_{n-1}$, we obtain $V_0^n \subset F_{n+1}$.

Now, each F_n is a well defined 2 to 1 pull back of F_{n-1}, so a cell C belongs to F_n if and only if its symmetric $-C \in F_n$. Finally, Part (5) follows in a similar manner to the analogous property of V_0^0 inside $Y_0^{(0)}$. □

3.2. **Frame labels.** Our next objective is to introduce a labeling system for pieces of the frame. This will allow us to describe the relative position of pieces of the nest within a central piece of the previous level. Unlike the case of unimodal maps, where nest pieces are always located left or right of the critical point, the possible labels for vertices of $\Gamma(F_n)$ will depend on the combinatorics of the critical orbit. Only after determining the labeling, it becomes possible to describe the location of nest pieces in a systematic manner.

Observe that the structure of F_{n+1} is trivially determined once we know F_n and the location of $g_n(0)$. A graphic way of seeing this is as follows. Say that the first return $g_{n-1}(0)$ to V_0^{n-2} falls in a cell $X \in F_n$. Let L_n and R_n be two copies of $\Gamma(F_n)$ with disjoint embeddings in the plane. Now connect L_n and R_n with a curve γ that does not intersect either graph. Suppose that one extreme of γ lands at the vertex of L_n that corresponds to X and the other extreme lands at the corresponding vertex of R_n *approaching it from the same access.*

Lemma 3.4. *If γ is collapsed by a homotopy of the whole ensemble, the resulting graph is isomorphic to $\Gamma(F_{n+1})$.*

Note: The above construction provides $\Gamma(F_{n+1})$ with a natural plane embedding; see lemma 3.5 below.

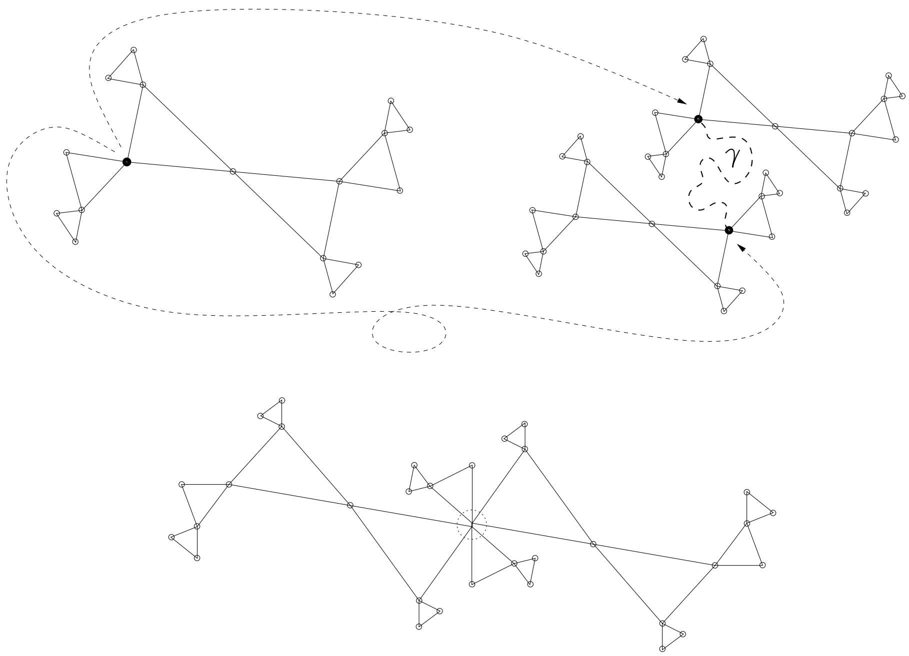

FIGURE 6. *The curve γ joins two copies of the same frame graph approaching the selected vertex from the same direction. The new frame graph is obtained after γ is contracted to a point.*

A label at level n will be a chain of $n+1$ symbols taken from the alphabet { 0,1, ..., (q-1), l, r, e, b, t }. First, put the labels { '0', '1', ..., '(q-1)' } on the cells of F_0, starting at the central piece $Y_0^{(0)}$ and moving counterclockwise.

Let σ_0 be the label of the cell that holds the first return of 0 to F_0 and, in general, let σ_n denote the label of the cell in $\Gamma(F_n)$ that holds the first return of 0. In order to label $\Gamma(F_{n+1})$, assume that we know the number q of pieces in F_0, and the *label sequence* $(q; \sigma_0, \dots, \sigma_{n-1})$ that identify the location of first returns of 0 to levels $0, \dots, n-1$ of the nest. In particular, all frames up to $\Gamma(F_n)$ have been successfully labeled.

Duplicate in L_n the labels of $\Gamma(F_n)$, but concatenate an extra 'l' at the beginning. Do a similar labeling on R_n by concatenating an extra 'r' to the duplicated labels. Note that the labels of the two vertices corresponding to X are $'\mathsf{l}'\sigma_n$ and $'\mathsf{r}'\sigma_n$. The labels on $\Gamma(F_{n+1})$ will be the same as those in the union of L_n and R_n except that we change the label of the identified vertex, to become $'\mathsf{0}'\sigma_n$.

Note: The above procedure does not give labels to the additional cells of F_2 that do not come from a pull back. These are the cells that are not drawn in heavy line in Figure 5 Being cells of level 2, their labels should have 3 symbols for consistency with the rest. The easiest way to do this is simply to impose the labels 'et1', 'et2', ..., 'et(q-1)' and 'eb1' ,'eb2', ..., 'eb(q-1)' in their natural order in the plane ('et' stands for **e**xtra piece on **t**op and 'eb' for **e**xtra piece on **b**ottom), then extend the labeling to higher levels as described.

Clearly, f induces a map $f_* : \Gamma(F_{n+1}) \longrightarrow \Gamma_n$ for $n \geq 2$, that acts by forgetting the leftmost symbol of each label. This is the case also for the induced map on the temporary frame F_2'.

3.3. **Properties of frame labellings.** Under certain conditions, label sequences give a complete characterization of the entire combinatorial structure. This is the content of Theorem 3.6. Before stating it, we need to review some properties of the frame and its labels.

Lemma 3.5. *The plane embedding of* Γ *does not depend on the homotopy class of the curve* γ *in lemma 3.4.*

Proof: Since we regard $\Gamma = \Gamma(F_n)$ as embedded in the sphere, the exterior of Γ is simply connected, so there is a natural cyclic order of accesses to vertices (some vertices can be accessed from more than one direction). In this order, all accesses to L_n are grouped together, followed by the accesses to R_n. $\square$

It is important to mention that the resulting labeling of $\Gamma(F_n)$ **does** depend on the access to ξ_n approached by γ. However, the final unlabeled graphs are equivalent as embedded in the plane.

As we just mentioned, some vertices are accessible from ∞ in two or more directions. These are precisely the vertices whose label contains the symbol '0' (for $n \geq 1$). Since such a vertex represents a frame cell that maps (eventually) to a central frame cell, the tail of a label with '0' at position j must be σ_j. On the other hand, for every j there must be labels with a '0' in position j. It follows that the set of labels of $\Gamma(F_n)$ and the sequence $(q; \sigma_0, \dots, \sigma_n)$ can be recovered from each other.

3.4. **Frames and nest together.** The definition of frame system was conceived to satisfy the properties of Proposition 3.3. An extension of the argument used to prove those properties shows that every piece V_j^n of the nest is contained in a frame cell of level $n+1$. Moreover, we would like to extend the definition of frames so that each V_j^n can be partitioned by a pull back of an adequate central frame. For this, we must recall first that $g_{n,j}(V_j^n) = V_0^{n-1} \supset F_{n+1}$.

Definition: The frame $F_{n,k}$ is the collection of pieces inside V_k^{n-2} obtained by the $g_{n-2,k}$-pull back of F_{n-1}. Elements of the frame $F_{n,k}$ are called **cells** and we will write $F_{n,0}$ instead of F_n, when there is a need to stress that a property holds in $F_{n,k}$ for every k.

If a puzzle piece A is contained in a cell $B \in F_{n,k}$, we denote B by $\Phi_{n,k}(A)$.

We have described already how to label F_n. The other frames $F_{n,k}$ $(k \geq 1)$, mapping univalently onto F_{n-1}, have a natural labeling induced from that of F_{n-1} by the corresponding $g_{n-2,k}$-pull back.

Let us describe now the itinerary of a piece V_j^n. Since $V_j^n \subset V_0^{n-1}$, the map g_{n-1} takes V_j^n inside some piece $V_{k_1(j)}^{n-1} \subset V_0^{n-2}$. Then, $g_{n-1,k_1(j)}$ takes $g_{n-1}(V_j^n)$ inside a new piece $V_{k_2(j)}^{n-1}$ and so on, until the composition of returns of level $n-1$

$$(g_{n-1,k_r(j)} \circ \dots \circ g_{n-1,k_1(j)} \circ g_{n-1})_{|V_j^n}$$

is exactly $g_{n,j} : V_j^n \mapsto V_0^{n-1}$. Of course, k_r is just 0, and we will write it accordingly.

We have extra information that deems this description more accurate. For the sake of typographical clarity, we will write k_i instead of $k_i(j)$. For $i \leq r$, let Φ_{n+1,k_i} be the cell in $F_{n+1,k_i} \subset V_{k_i}^{n-1}$ that contains

$$g_{n-1,k_i} \circ \dots \circ g_{n-1,k_1} \circ g_{n-1}(V_j^n)$$

and denote by λ_{n+1,k_i} the label of Φ_{n+1,k_i}.

Definition: The **itinerary** of V_j^n is the list of piece-label pairs:

$$\chi(V_j^n) = \Big([V_{k_1}^{n-1}; \lambda_{n+1,k_1}], [V_{k_2}^{n-1}; \lambda_{n+1,k_2}], \dots, [V_{k_{r-1}}^{n-1}; \lambda_{n+1,k_{r-1}}], [V_0^{n-1}; \lambda_{n+1,0}]\Big) \tag{3.1}$$

up to the moment when V_j^n maps onto V_0^{n-1}.

Note first of all that the last label, $\lambda_{n+1,0}$, will start with '0' due to the fact that V_0^{n-1} is in the central cell of F_n. More importantly, the conditions

$$\begin{array}{rcll} V_{k_1}^{n-1} & \subset & g_{n-1}(\Phi_{n+1,0}) & \\ V_{k_{i+1}}^{n-1} & \subset & g_{n-1,k_i}(\Phi_{n+1,k_i}) & 2 \leq i < r \end{array} \tag{3.2}$$

must hold since we know that $g_{n-1,k_{i-1}} \circ \ldots \circ g_{n-1,k_1} \circ g_{n-1}(V_j^n) \subset \Phi_{n+1,k_i}$ and $g_{n-1,k_i} \circ \ldots \circ g_{n-1,k_1} \circ g_{n-1}(V_j^n) \subset V_{k_{i+1}}^{n-1}$.

Definition: When we specify the sequence of frame labellings up to a given level n, the locations of the nest pieces and their (admissible) itineraries, we say that we have described the **combinatorial type** of the map at level n. If $|\mathcal{V}^n| < \infty$ we say that the type is **finite**; refer to Lemma 2.6 and Definition 2.6.

Condition 3.2 will be called the **frame admissibility condition**.

3.5. **Real frames.** Let us digress momentarily in order to compare the above definitions with their counterparts in the real case.

When the parameter c is real, all the pieces of the nest intersect the real axis. Call I_j^n the intersection of V_j^n with $\mathbb{R}$. The combinatorial type of the nest is determined by how many intervals are there left and right of I_0^n, the sign (orientation) of each map $g_{n,j} : I_j^n \longrightarrow I_0^{n-1}$ and the itineraries of all I_j^n through intervals of the previous level. If we specify an arbitrary type, the *unimodal admissibility conditions* are necessary so that the type can be realized; these conditions require

- Since $g_{n-1,k}$ is supposed to take I_k^{n-1} onto I_0^{n-2}, the order of the intervals inside I_k^{n-1} is preserved or reversed according to the orientation of $g_{n-1,k}$.
- Since $g_{n,j} : I_j^n \longrightarrow I_0^{n-1}$ is supposed to be the composition of all g_{n-1,k_i} specified by the itinerary of I_j^n, the sign of $g_{n,j}$ must be the product of signs of the g_{n-1,k_i} when I_j^n is right of I_0^n and the negative of that sign when I_j^n is to the left of I_0^n (or the other way, if $g_{n,0}$ reverses orientation).

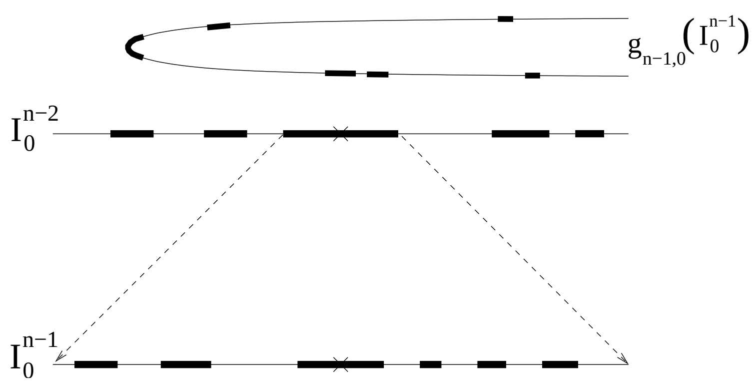

FIGURE 7. *Illustration of the unimodal admissibility conditions. The map $g_{n-1,0}$ spreads the intervals of level n inside some intervals of level $n-1$. However, the order of the right intervals is respected and that of the left intervals is reversed. Note that the orientation of each left interval is also reversed and that I_0^n maps to the leftmost position.*

We note first that both conditions emphasize the fact that $g_{n,0}$ is unimodal. The first map $g_{n-1,0}$ can mix left intervals with right intervals as in Figure 7, but the order of the right intervals is preserved and the order of the left ones is reversed (or vice-versa). The second condition specifies that the orientation of each $g_{n,j}$ is the product of the orientations of all intermediate steps *including the fact that $g_{n-1,0}$ has different orientations on each side of 0.* The important observation to make is that the simplicity of the unimodal admissibility conditions is due to the existence of a natural order on $\mathbb{R}$. In the more general case of complex polynomials, the order of intervals is replaced

by relative locations of nest pieces within a frame. The requirement that relative orders are preserved is replaced by Conditions 3.2 and the rule of signs is replaced by a compatible choice of labels.

3.6. **Combinatorial classification.** We are ready to state the main theorem of this Section. In loose language, it states the existence within the quadratic family, of arbitrary admissible finite combinatorial types.

Definition: We will say that two non-renormalizable polynomials are **weakly combinatorially equivalent** if they have the same combinatorial types at every level, so that they differ only by the orientation of their frames.

Note: The point $g_n(0)$ is contained in V_0^{n-1}. In particular, it is possible to apply the map g_{n-1} to it and, in fact, we could keep composing first return maps of lower levels until the first return of the critical orbit to V_0^n. This argument shows that for weak combinatorially equivalent maps, g_{n+1} is formed by the same composition of previous levels first return maps and consequently, *the first returns to corresponding pieces happen at the same times.* In the next sections we will make use of this property.

Theorem 3.6. *Consider a finite combinatorial type of level n, together with a parapiece Δ of parameters that satisfy it up to level $n-1$. Let ℓ be the level of the last lateral return prior to level n and let*

$$r = \begin{cases} 1 & \text{if } g_n \text{ is a central return} \\ 2^{n-\ell} & \text{if } g_n \text{ is lateral.} \end{cases}$$

Then there exist r parapieces inside Δ each consisting of parameters satisfying the same weak combinatorial type to level n.

Moreover, for any such parapiece Δ', the first returns $\{g_n[c] \mid c \in \Delta'\}$ form a full DH quadratic-like family.

Note: This property of accumulating powers of 2 during central cascades is related to the phenomenon that makes Lyubich's theorem possible. Namely, the fact that the moduli grow linearly from lateral return to lateral return, even though they decrease by half on each central return.

Proof: We are already acquainted with the central symmetry of frames. It is obvious that the dual graph of a frame can be symmetric only about its critical vertex ξ. Because of this, the frame $F_{\ell+1}$ cannot be symmetric around the lateral cell C where $g_\ell(0) \in \left(V_0^{\ell-1} \setminus V_0^\ell\right)$ falls, so the pull back $F_{\ell+2}$ cannot have more than 2-fold symmetry around the origin.

By definition, the (possibly empty) sequence $\{g_{\ell+1}, \ldots, g_{n-1}\}$ is the beginning of a cascade of central returns of length $n-\ell$. Therefore, the frame graph $\Gamma(F_{n+2})$ has exactly $(2^{n-\ell})$-fold symmetry around ξ_n.

Let $c \in \Delta$. Every map $g_{n-1,k}$ takes its corresponding piece V_k^{n-1} onto V_0^{n-2}. Then the pull back by $g_{n-1,k}$ of any region inside V_0^{n-2} is well defined and located inside V_k^{n-1}. In particular, for every piece V_j^n listed in the type of level n, the itinerary prescribes the sequence of returns $g_{n-1,0}, g_{n-1,k_1}, \ldots, g_{n-1,k_r}$, so the univalent pull back of V_0^{n-1} under the composition $(g_{n-1,k_1} \circ \ldots \circ g_{n-1,k_r})$ is a well defined piece inside $V_{k_1}^{n-1}$. Let us name this piece U_j'.

Clearly $U_0' \subset V_1^{n-1}$ because the itinerary of the critical piece V_0^n begins with the first return of 0 to level $n-1$. As c moves within Δ, this return can be made to fall in U_0'. All c with this property form a parapiece $\Delta^* \Subset \Delta$ that can be described as the set of parameters for which the itinerary of U_0 is as originally prescribed; i.e. $U_0 = V_0^n$. For the rest of the argument we will restrict c to Δ^*.

For $j \geq 1$, the $g_{n-1,0}$-pull back of U_j' will be called U_j; however, $g_{n-1,0}$ is 2 to 1, so we have to decide on a frame orientation before locating these pieces inside F_{n+2}.

The combinatorial type of level n involves the label σ_{n+2} that specifies the cell in F_{n+2} containing the first return $g_n(0)$. If this return is central there is no choice: The return falls on the piece V_0^{n+1} inside the central cell. Otherwise, we need to recall the discussion above. After a (possibly vacuous) cascade of central returns, there are $\frac{r}{2} = 2^{n-\ell-1}$ cells of F_{n+1,k_1} that can be labeled with σ_{n+2} and contain U_1'. This comes from the $n-\ell-1$ choices of orientation taken from level $\ell+1$ to $n-1$. Assuming that the return $g_n(0)$ is lateral, there is one more choice of orientation to make, so F_{n+2} has $(2^{n-\ell})$ cells that can host U_j. Once this decision is made, the label orientation is determined and the rest of the pieces U_j are forcibly placed around the frame F_{n+2}.

We have constructed pieces $U_j \subset V_0^{n-1}$ that follow the given itineraries. It rests now to show that for some parameters $c \in \Delta^*$, the U_j can be made to coincide with the respective V_j^n. This can be shown as follows. The itinerary of V_0^n (and of 0) ends with the first return g_n of 0 to V_0^{n-1}. This return generates a full family for $c \in \Delta^*$, so we can choose a parapiece Δ^{**} of c such that $g_n(0) \in U_1$.

The second return to V_0^{n-1} is specified by the itinerary of U_1. From this observation we conclude that $U_1 = V_1^n$ from the definition of nest. Also, this second return generates a full family for $c \in \Delta^{**}$, so we can choose an even smaller parapiece Δ^{***} of parameters c such that $g_{n,1}(0) \in U_2$. This argument can be pursued till the end to obtain the parapiece Δ' of values c for which every $U_j = V_j^n$. □

Repeated application of Theorem 3.6 yields the following.

Corollary 3.7. *Arbitrary infinite sequences of finite, weak combinatorial types can be realized in the quadratic family, as long as they satisfy the admissibility condition at every level. The set of parameters satisfying the complete type is the intersection of a family of nested sequences of parapieces, with 2^n of them at every non-central level n.*

Proof: This is clear, since each Δ contains at least one parapiece Δ' that satisfies the combinatorial type at level n. The collection of first return maps of level n for parameters in Δ' forms a full family, so we can apply Theorem 3.6 again. An arbitrary choice of orientation at every level gives an infinite nested sequence of parapieces. Evidently, a parameter in the intersection satisfies the prescribed combinatorics at every level.

Every level accounts for one dyadic choice of orientation. Although they are not apparent during central cascades, the previous proof shows that they accumulate to display $2^{n-\ell}$ pieces of level n inside each of the 2^ℓ pieces of (lateral) level ℓ. □

The set of parameters that are combinatorially equivalent to a given one cannot be completely characterized without some amount of analytical information. Corollary 3.7 describes such set as a collection of nested sequences of parametric pieces, but it does not say whether they intersect in single points or in more complicated regions. The fact that the parapieces shrink to a unique parameter amounts to *combinatorial rigidity*; this was the strategy of Yoccoz to establish local connectivity in the case of non-renormalizable polynomials. For such parameters, he showed that the sum of paramoduli is infinite, so the set of parameters in the nested intersections of parapieces becomes a Cantor set. In particular, if the type includes no central returns, every parapiece contains exactly two pieces of the next level and the Cantor set has a natural dyadic structure. Thus, for some precise sequences of combinatorial types, the choice of frame orientations at every level may single out a unique parameter.

Note: It should be remarked that alternative classifications of combinatorial properties are possible and indeed quite useful. Of particular notice is D. Schleicher's concept of *internal addresses* (see [LS]), describing a combinatorial type in terms of an irreducible sequence of hyperbolic components that encodes the critical orbit information with increasing precision.

4. Examples

We present here two instances of the use of our combinatorial model. Every first renormalization type corresponds to a maximal hyperbolic component of the Mandelbrot set; these are classified in 4.1. A rotation-like map is an unimodal map whose postcritical set is semi-conjugate to a circle rotation; the Fibonacci map being n instance. In 4.2 we find complex quadratic maps with the same property. Other applications, including a classification of complex quadratic Fibonacci polynomials, can be found in [P].

4.1. **Maximal hyperbolic components.** Consider an arbitrary combinatorial type up to some level n, with the property that the last return is not central. Upon specifying a frame orientation, there is a unique parapiece Δ consisting of parameters that satisfy the given combinatorics. Clearly, parapieces corresponding to different types must be disjoint.

When the return to level $n+1$ is central, there is no need to orient the frame; that is, there is a unique piece $\Delta' \subset \Delta$ of parameters featuring this central return. Then, if a parameter in Δ has an infinite cascade of central returns starting at level $n+1$, its combinatorial type will be completely determined by the initial n levels. The unique sequence of nested parapieces $\Delta \supset \Delta' \supset \ldots$ intersects in the set M' of renormalizable parameters whose first n nest levels are as prescribed. It is known that M' is quasi-conformally homeomorphic to M (see [DH1] and [L3]). In fact, this homeomorphism is given by **straightening**: For every $c \in M'$ there is a quasi-conformal map h that realizes the conjugation $h \circ g_n = f \circ h$ between $g_n[c]$ and some quadratic polynomial f; moreover, h satisfies $\overline{\partial} h = 0$ on the small filled Julia set of $g_n[c]$.

Since the parameters in M' have a well defined nest, the renormalization is not of immediate type. The base of such "small copy" of M is a primitive hyperbolic component H. Since H is a quasi-conformal deformation of $\heartsuit$, its boundary has a cusp point. Also, the parameters in H are exactly once renormalizable, so H is maximal (see definitions at the beginning of Section 2).

The above discussion shows that any finite frame type is associated to a maximal hyperbolic component of M. Conversely, each maximal copy of M is encoded by the type of its frame, that is, by the associated graph $\Gamma(F_{n+1})$ or its label sequence. Note that the frame graph of level $n' > n$ consists of a bouquet of $2^{n'-n}$ copies of $\Gamma(F_{n+1})$ with their central vertices identified. This is illustrated in the right hand example in Figure 5. The beautiful pictures of small Mandelbrot copies with hundreds of mini-copies spiraling in all directions belong naturally to the class of finite nest types that conclude with a long central cascade.

4.2. **Rotation-like maps.** Let $c_{\text{fib}} = -1.8705286321\ldots$ parametrize the Fibonacci map $z \mapsto z^2 + c_{\text{fib}}$. This is the unique real quadratic polynomial with the property that the critical orbit has closest returns to 0 exactly when the iterates are the Fibonacci numbers; see [LM]. In terms of the principal nest, $f_{c_{\text{fib}}}$ satisfies the equivalent condition:

> For $n \geq 2$, each level of the principal nest consists of the central piece V_0^n and a unique lateral piece V_1^n. The first return map of previous level $g_{n-1} : V_0^{n-1} \longrightarrow V_0^{n-2}$ interchanges the central and lateral roles:
> $$g_{n-1}(V_0^n) \Subset V_1^{n-1}, g_{n-1}(V_1^n) = V_0^{n-1}.$$
> Additionally, the first returns to $Y_0^{(1)}$ and V_0^0 happen on the third and fifth iterates respectively.

To discern the critical orbit behavior of $f_{c_{\text{fib}}}$, note that every level of the nest has a unique lateral piece and so, in a sense, every first return comes as close as possible to being central without actually being so. This means that the map $f_{c_{\text{fib}}}$ is not renormalizable in the classical sense, although its

combinatorics can be described as an infinite cascade of *Fibonacci renormalizations* in the space of **gql** maps with one lateral piece.

The Fibonacci map features as a decisive case in the proof of Lyubich's theorem; see [L2]. Here we will describe a family of unimodal maps with similar behavior and extend it to a family of complex quadratic maps.

Let $S = (S_0, S_1, \ldots)$ be a strictly increasing sequence of numbers such that $\frac{S_{j+1}}{S_j} \leq 2$. The S-*odometer* is a symbolic dynamical system (Ω, T) defined as follows. For any nonnegative n there is a k such that $S_k \leq n < S_{k+1}$. Then $n = S_k + n_1$ with $n_1 < S_k$. By splitting further $n_1 = S_{k'} + n_2$ (with $k' < k$ and $n_2 < S_{k'}$) and so on, we obtain the decomposition

$$n = d_k \cdot S_k + \ldots + d_0 \cdot S_0$$

where each d_j is either 0 or 1. Letting $d_j = 0$ for $j > k$, we get the sequence

$$\langle n \rangle = (d_0, d_1, \ldots) \in \{0,1\}^{\mathbb{N}}.$$

We use $\langle \mathbb{N} \rangle$ to denote $\{\langle n \rangle \mid n \in \mathbb{N}\}$ and let Ω be the closure

$$\Omega = \overline{\langle \mathbb{N} \rangle} = \{\omega \in \{0,1\}^{\mathbb{N}} \mid \sum_{i=0}^{j} \omega_j S_i < S_{j+1} \text{ for all } j \geq 0\}.$$

The map $T : \langle \mathbb{N} \rangle \longrightarrow \langle \mathbb{N} \rangle$ is given by $T\langle n \rangle = \langle n+1 \rangle$. This map does not always extend uniquely to Ω. When there is an extension, the dynamical system (Ω, T) obtained from the sequence S is called a S-**odometer**. It can be described as an adding machine with variable stepsize.

Let us relate the above concept to interval dynamics. First, some definitions.

Consider a unimodal map $f : I \longrightarrow I$ where $I = [c_1, c_2]$ and $\{0, c_1, c_2, \ldots\}$ is the critical orbit. Let $D_1 = [c_1, 0]$ and, for $n \geq 2$, define

$$D_{n+1} = \begin{cases} [c_{n+1}, c_1] & 0 \in D_n \\ f(D_n) & 0 \notin D_n \end{cases}$$

The sequence $S = (S_0, S_1, \ldots)$ of *cutting times* consists of those n such that $0 \in D_n$. Note that $S_0 = 1$. It is easy to show that $S_{k+1} - S_k$ is also a cutting time so we can define the **kneading map** $Q : \mathbb{N} \longrightarrow \mathbb{N}$ by the relation

$$S_{Q(k)} = S_{k+1} - S_k.$$

Lemma 4.1. *If S is the sequence of cutting times of a unimodal map f, the following characterization of Ω holds:*

$$\Omega = \{\omega \in \{0,1\}^{\mathbb{N}} \mid \omega_j = 1 \Rightarrow \omega_i = 0 \text{ for } Q(j+1) \leq i \leq j-1\}.$$

Also, if $Q(k) \longrightarrow \infty$, then T extends uniquely to Ω and is conjugate to the action of f on its postcritical set.

See [BKP] for proofs.

In the case of the Fibonacci polynomial, the above definitions correspond to the description of the critical orbit in Subsection 3 of [LM]. There it is shown that $(\Omega, T)_{c_{\mathrm{fib}}}$ is semiconjugate to the circle rotation by $\rho = \frac{\sqrt{5}-1}{2}$. Real **rotation-like maps**, as defined in [BKP], are unimodal maps that generalize this behavior.

Let $\rho \in [0,1) \setminus \mathbb{Q}$ with continued fraction expansion $\rho = [a_1, a_2, \ldots]$ and denote its convergents with $\frac{p_i}{q_i}$ so that $\frac{p_0}{q_0} = \frac{0}{1}$ and $\frac{p_1}{q_1} = \frac{1}{a_1}$.

Theorem 4.2. [BKP] *Consider the sequence r_k starting with $r_1 = q_1 - 1$ and whose $(k+1)^{st}$ element is given recursively by $r_{k+1} = r_k + a_{k+1}$. Then the S-sequence given by*

$$\begin{aligned} S_{r_k} &= q_k \\ S_{r_k+j} &= (j+1)q_k \quad \text{for } 1 \le j < a_{k+1} \end{aligned}$$

is realized as the sequence of cutting times of some quadratic polynomial. Moreover, the application

$$\Pi_\rho(\omega) = \sum \omega_j S_j \rho \,(mod\, 1)$$

from Ω to the unit circle is well defined and continuous. This map satisfies $\Pi_\rho \circ T = R_\rho \circ \Pi_\rho$, where R_ρ is the rotation by angle ρ, and is 1 to 1 everywhere except at the preimages of 0.

In terms of the principal nest, the behavior that characterizes rotation-like maps is a succession of central cascades followed by one lateral escape. That is, the critical orbit falls in $V_0^{S_k-1}$ starting a central cascade. After iterating the first return map g_k for $a_k - 1$ turns, we get a lateral return on $V_1^{S_k}$. Next, $g_{S_k,1}$ creates a new cascade and so on. In particular, the Fibonacci map is the special case of a rotation-like map where every central cascade has length 0.

Consider an arbitrary sequence $a_1, a_2, \ldots$ of positive integers. We will construct now a Cantor set of *complex* rotation-like parameters with central cascades of length $a_i - 1$. By theorem 3.6, it is only necessary to give an admissible description of labeling sequences and to show that it models the combinatorics mentioned above.

The initial labeling data for our map is $q = 2$ and $\sigma_0 = '1'$, so rotation-like maps will all be located in the 1/2-limb. Note also that on central return levels, $\sigma_{k+1} = '0'\sigma_k$. Therefore, we only need to specify the labels σ_{r_η} for $r_\eta = \sum a_j$.

Let $(\tau_1, \tau_2, \ldots)$ be a sequence of random chains of 'l''s and 'r''s so that τ_i has length $a_i + 1$. Set $\sigma_{r_1} = \tau_1'0'$ and $\sigma_{r_2} = \tau_2\sigma_{r_1-1} = \tau_2'00\ldots01'$. Now we can define inductively $\sigma_{r_j} = \tau_j'0'\sigma_{r_{j-1}-1}$.

Proposition 4.3. *The label sequence $(q; \sigma_0, \sigma_1, \ldots)$ defined above is admissible, it completely describes a combinatorial type and the corresponding map is rotation-like.*

Proof: The fact that the sequence of labels determines the type can be seen to be true since there are no consecutive lateral returns. This implies that the nest has exactly one lateral piece at those levels (and none elsewhere) so its position within the frame is completely determined by σ_{r_j}.

As mentioned above, $'0'\sigma_k$ (when $k \neq r_j$) is an admissible label since it corresponds to the central cell of F_{k+1}. Now consider what happens to the central cell labeled $'0'\sigma_{r_{j-1}-1}$. Since level r_{j-1} corresponds to a non-central return, $F_{r_{j-1}+1}$ has two preimages of that cell, labeled $'l0'\sigma_{r_{j-1}-1}$ and $'r0'\sigma_{r_{j-1}-1}$ respectively. On consecutive central returns, we double the number of pull-backs of such cells and thus, use all possible combinations of $'l'$ and $'r'$ to label them. A glance to the frame graph shows that these are the cells neighboring the central one (see [Sm]). An eventual lateral return must fall precisely in one of these cells, and this is what happens when $\sigma_{r_j} = \tau_j'0'\sigma_{r_{j-1}-1}$. □

The real rotation-like maps studied in [BKP] correspond to a careful choice of the τ_j. In fact, it is possible to extract a kneading sequence from the rotation number data. Then, a result of Yoccoz guarantees that there is a unique real polynomial in that combinatorial class.

The complex maps corresponding to other choices of τ_j's have the same weak combinatorial behavior, so the critical orbits of two maps with the same sequence $a_1, a_2, \ldots$ are conjugate. In particular we obtain the following result.

Corollary 4.4. *Given the sequence $a_1, a_2, \ldots$ there exists an infinite family of complex quadratic polynomials for which the postcritical set is conjugate to an S-odometer and semi-conjugate to the circle rotation of angle $\rho = [a_1, a_2, \ldots]$.*

Appendix

A. **Holomorphic motions of puzzle pieces and winding number.** Consider the following

Definition: Let $X_* \subset \overline{\mathbb{C}}$ be an arbitrary set and $\Delta \subset \mathbb{C}$ a simply connected domain with $*$ as a base point. A **holomorphic motion** of X_* over Δ is a family of injections $h_\lambda : X_* \longrightarrow \overline{\mathbb{C}}$ $(\lambda \in \Delta)$ such that for each fixed $x \in X_*$, $h_\lambda(x)$ is a holomorphic function of λ and $h_* = \mathrm{id}$. For every $\lambda \in \Delta$ we write X_λ to denote the set $h_\lambda(X_*)$.

Holomorphic motions are extremely versatile owing to their regularity properties. The motion can always be extended beyond X_* and is transversally quasi-conformal. This is the content of the *λ-lemma*.

Theorem A.1. [Sl], [MSS] (**the λ-lemma**) *For every holomorphic motion $h_\lambda : X_* \longrightarrow \overline{\mathbb{C}}$, there is an extension to a holomorphic motion $H_\lambda : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$. The extension to the closure $\overline{h}_\lambda : \overline{X}_* \longrightarrow \overline{\mathbb{C}}$ is unique. Moreover, there is a function $K(r)$ approaching 1 as $r \to 0$ such that the maps h_λ are $K(r)$-quasi-conformal, where $r = d_\Delta(*, \lambda)$ is the hyperbolic distance between $*$ and λ in Δ.*

We are interested in the case when the holomorphic motion is defined over a parapiece Δ of M. In agreement with the notation used in the main body of this work, we use c instead of the classical λ to denote parameters in Δ. When an object is defined for any $c \in \Delta$, we express its dependence on the parameter by writing OBJ$[c]$.

As mentioned in Section 2, Δ can be interpreted as the set of parameters for which *a given combinatorial behavior holds, up to a return $g(0)$ of the critical orbit to some central piece V.* In particular, this description provides a natural base point for Δ. Namely, the superattracting parameter c_0 for which $g_{c_0}(0) = 0$. The little M-copy associated to Δ can be defined as the set of parameters for which the iterates $\{g_c(0), g_c^{\circ 2}(0), \dots\}$ remain in $V[c]$ (refer to Subsection 4.1).

The dynamics in the region N_c (defined at the beginning of Subsection 2.2) is always conjugate to $z \mapsto z^2$, so varying the parameter $c \in \mathbb{C}$ provides a holomorphic motion of any specified (open) ray or equipotential. When c is restricted to Δ, the combinatorics require that some rays land together, enclosing the boundary of $V[c]$. Since the intersection $\partial V \cap K$ is a collection of preimages of the fixed point α and these vary holomorphically with c, there is a natural holomorphic motion of $\partial V[c_0]$ over Δ. This can be extended to a holomorphic motion $h_c : V[c_0] \longrightarrow V[c]$.

The holomorphic motion of a puzzle piece can be viewed as a complex 1-dimensional foliation of the bi-disk

$$\mathbb{V} = \bigcup_{c \in \Delta} V[c] \in \mathbb{C}^2$$

whose leaves are the graphs of the functions $c \mapsto h_c(p)$ for every $p \in V[c_0]$. Under this interpretation we will write $\{c \mapsto V[c] \mid c \in \Delta\}$ to refer to the motion.

Definition: A correspondence $c \mapsto \phi(c)$ such that $\phi(c) \in V[c]$ determines a section $\phi : \Delta \longrightarrow \mathbb{V}$ of the holomorphic motion h. It is said to be a **proper holomorphic section** if it maps $\partial\Delta$ into the torus $\delta\mathbb{V} = \bigcup_{c \in \partial\Delta} \partial V[c]$.

We say that a proper section $\{c \mapsto \phi(c)\}$ has **winding number** n if the curve $\phi(\partial\Delta)$ has winding number n with respect to the vertical generator of the 1-dimensional homology of $\delta\mathbb{V}$.

In the case $\phi(c) = g_c(0)$, this return map determines a proper section since $g_c(0) \in V[c]$ for all c and $c \in \partial\Delta \Rightarrow g_c(0) \in \partial V[c]$. Each return map $g_c : g_c^{-1}(V) \longrightarrow V$ is a quadratic-like map and the associated map

$$g_c : \mathbb{U} \longrightarrow \mathbb{V},$$

where $\mathbb{U} = \bigcup g_c^{-1}(V[c])$, is called a **DH quadratic-like family**. We can interpret intuitively the fact that a family has winding number n as saying that, as c goes once along $\partial\Delta$, the point $g_c(0)$ goes n times around the (moving) boundary of the piece $V[c]$.

An immediate consequence of extending the holomorphic motion of $\partial V[c_0]$, is the fact that $\{g_c \mid c \in \Delta\}$ is a full family; that is, there is a homeomorphism $\mathrm{Hyb} : \widetilde{M} \longrightarrow \Delta$ from a neighborhood $\widetilde{M}$ of M to Δ with the following property: For every parameter $c' \in \widetilde{M}$, $g_{\mathrm{Hyb}(c')}$ is hybrid equivalent[2] to $z \mapsto z^2 + c'$. This of course, justifies the existence of the small M-copy associated to Δ.

References

[BH1] B. Branner, J. Hubbard. *The iteration of cubic polynomials. Part I: The global topology of parameter space.* Acta Math., **160** (1988), 143-206.

[BH2] B. Branner, J. Hubbard. *The iteration of cubic polynomials. Part II: Patterns and parapatterns.* Acta Math., **169** (1992), 229-325.

[BKP] H. Bruin, G. Keller and M. st. Pierre. *Adding machines and wild attractors.* Ergod. Th. & Dynam. Sys., **17** (1997), 1267-1287.

[D2] A. Douady. *Chirurgie sur les applications holomorphes.* In: Proc. ICM, Berkeley, (1986), 724-738.

[DH1] A. Douady & J. H. Hubbard, *Étude dynamique des polynômes complexes I & II.* Publ. Math. Orsay, 1984-85.

[DH2] A. Douady & J. H. Hubbard, *On the dynamics of polynomial-like maps.* Ann. Sci. Éc. Norm. Sup., **18** (1985), 287-343.

[GLT] P. J. Grabner, P. Liardet and R. F. Tichy. *Odometers and systems of enumeration.* Acta Arithmetica, **70** (1995), 103-123.

[H] J. H. Hubbard, *Local connectivity of Julia sets and bifurcation loci: Three theorems of J.-C. Yoccoz.* In: Topological Methods in Modern Mathematics pp. 467-511 (ed. L. Goldberg & A. Phillips), (Publish or Perish, 1993).

[J] W.Jung. PC software `mandel.exe`; available at: `http://www.iram.rwth-aachen.de/~jung/indexp.html`

[LS] E. Lau and D. Schleicher. *Internal Addresses of the Mandelbrot Set and Irreducibility of Polynomials.* Preprint IMS at Stony Brook, # 1994/19.

[L1] M. Lyubich. *Combinatorics, geometry and attractors of quasi-quadratic maps.* Ann. of Math., **140**, (1994), 347-404.

[L2] M. Lyubich. *Dynamics of quadratic polynomials, I-II.* Acta Math., **178** (1997), 185-297.

[L3] M. Lyubich. *Dynamics of quadratic polynomials, III. Parapuzzle and SBR measures.* In: Géométrie Complexe et Systémes Dynamiques. Volume in Honor of Adrien Douady's 60th Birthday. Astérisque 261, (2000), 173-200.

[LM] M. Lyubich and J. Milnor. *The Fibonacci unimodal map.* J. Amer. Math Soc., **6** (1993), 425-457.

[MSS] R. Mañé, P. Sad and D. Sullivan. *On the dynamics of rational maps.* Ann. Sci. Éc. Norm. Sup., **4**, (1983), 193-217.

[Ma] M. Martens. *Distortion results and invariant Cantor sets of unimodal maps.* Ergod. Th. & Dynam. Sys., **14**, (1994), 331-349.

[M1] J. W. Milnor, Dynamics in One Complex Variable. (Vieweg, 1999).

[M2] J. W. Milnor, *Periodic orbits, external rays and the Mandelbrot set: An expository account.* In: Asterisque 261 'Geometrie Complexe et Systemes Dynamiques', pp. 277-333, (SMF 2000).

[P] R. Pérez, *Geometry of Q-recurrent maps.* In preparation.

[R] P. Roesch. *Holomorphic motions and puzzles (following Shishikura).* In: The Mandelbrot set, theme and variations, edited by Tan Lei, LNS 274, Cambridge, (2000).

[Sl] Z. Slodkowsky. *Holomorphic motions and polynomial hulls.* Proc. Amer. Math. Soc., **111**, (1991), 347-355.

[Sm] D. Smania. *Puzzle geometry and rigidity: The Fibonacci cycle is hyperbolic.* **arXiv**: `math.DS/0203164.`

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[2] see Subsection 4.1